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THE DIMENSION OF HYPERSPACES OF CONTINUA(General Topology,Dimension and Set Theory)

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THE DIMENSION OF HYPERSPACES OF CONTINUA

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1. Introduction.

By a *continuum* we mean a compact connected metric space. Let X be a continuum. The *hyperspaces* of a continuum X are the following.

$$2^X = \{A \mid A \text{ is a closed subset of } X \text{ and } A \neq \emptyset\} \text{ and}$$

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\}.$$

Let $A, B \in 2^X$. Define a metric $d_H(A, B)$ by

$d_H(A, B) = \inf\{\varepsilon > 0 \mid U(A, \varepsilon) \supset B \text{ and } U(B, \varepsilon) \supset A\}$, where $U(A, \varepsilon)$ denotes the ε -neighborhood of A in X .

Then the metric d_H is called the *Hausdorff metric*. Then 2^X and $C(X)$ are pathwise connected continua. In this note, we consider the dimension of 2^X and $C(X)$.

2. Dimension of 2^X and $C(X)$.

Mazurkiewics showed the following theorem about the dimension 2^X .

2.1. Theorem. Let X be a nondegenerate continuum. Then 2^X contains the Hilbert cube $Q = [0,1]^\omega$. Hence $\dim 2^X = \infty$.

By 2.1, the dimension of 2^X has been completely determined. Next, we discuss the dimension of $C(X)$.

2.2. Theorem (Kelley). If X is a Peano (locally connected) continuum, then $C(X)$ has a finite dimension if and only if X is a finite linear graph.

For the case of non Peano continua, we have

2.3. Theorem (Kelley). If X is a hereditarily indecomposable continuum with $\dim X \geq 2$, then $\dim C(X) = \infty$.

A continuum X is decomposable if there exist $A, B \in C(X)$ such that $A \cup B = X$ and $A \neq X \neq B$. A continuum X is hereditarily indecomposable if any subcontinuum of X is not decomposable. It is well-known that the pseudo-arc is a hereditarily indecomposable continuum.

2.4. Theorem (Bing). Let X be a continuum with $\dim X = n$. Then there exists a subcontinuum A of X such that A is hereditarily indecomposable and $\dim A = n - 1$.

Combining 2.3 and 2.4, we have the following

2.5. Theorem (Eberhart, Nadler and Rogers). *If X is a continuum with $\dim X \geq 3$, then $\dim C(X) = \infty$.*

Finally, the following interesting problem remains open.

Problem 1. If X is any 2-dimensional continuum, then must $\dim C(X) = \infty$?

In relation to Problem 1, some partial answers have been obtained.

2.6. Theorem (Rogers). *If X is an arcwise connected continuum and $\dim X = 2$, then $\dim C(X) = \infty$.*

If X contains a subcontinuum which is homeomorphic to a cartesian product of two (nondegenerate) continua, then $\dim C(X) = \infty$.

3. Dimension of certain 2-dimensional continua.

In this section, we give a partial answer to above problem.

Let D be the closed unit ball in the plane, $D = \{(x,y) \in E^2 \mid x^2 + y^2 \leq 1\}$, and let $S^1 = \{(x,y) \in D \mid x^2 + y^2 = 1\}$. A map f from a topological space Z to D is essential provided that

$$f|f^{-1}(S^1): f^{-1}(S^1) \rightarrow S^1$$

can not be extended to a map defined on all of Z to S^1 . A map $f: X \rightarrow Y$ between continua is *weakly confluent* provided that for each subcontinuum B of Y there is a subcontinuum A of X such that $f(A) = B$.

The following fact is very useful in continua theory and we need it.

3.1 (Mazurkiewicz). *Let X be a continuum. If a map $f: X \rightarrow D$ is essential, then f is weakly confluent.*

Let $H^1(X)$ denote the Čech cohomology group of X . Then we have

3.2. Theorem (Grispolakis, Tymchatyn and Kato). *If X is a 2-dimensional continuum and $\text{rank } H^1(X) < \infty$, then $\dim C(X) = \infty$. More precisely, $C(X)$ contains the Hilbert cube Q , hence $C(X)$ is strongly infinite-dimensional.*

The first part of 3.2 already has been obtained by combining some theorems of [20] and [9].

Recently, the author learned the existence of the paper [20]. But, the proof of author is different from that of [20] (see [5]) and it is more simple. In [5], We used the following theorem.

3.3. Theorem (Kato). *If X is a continuum with $\dim X \geq 2$, then there is an uncountable closed subset A of $C(X)$ such that if $A \in A$,*

then $\text{rank } H^1(A) = \infty$.

3.4. Corollary (Kato). *If X is a continuum and each subcontinuum is an FANR, then $\dim X \leq 1$.*

3.5. Remark. If there exists a continuum X with $\dim X = 2$ such that $\dim C(X) < \infty$, then the continuum X has a quite strange property. For example, for any closed subset A having countable components and any map $f: A \rightarrow S^1$, there is an extension $F: X \rightarrow S^1$ of f . It is well-known that a continuum Y is $\dim Y \geq 2$ if and only if there exist a closed subset A of Y and a map $f: A \rightarrow S^1$ which cannot be extended over Y . Also, it is known that if Z is a hereditarily indecomposable continuum, then for any closed subset A having countable components and any map $f: A \rightarrow S^1$ there is an extension $F: X \rightarrow S^1$ of f .

Let \mathcal{P} be a collection of compact polyhedra. A continuum X is \mathcal{P} -like if for any $\varepsilon > 0$ there is an onto map $f: X \rightarrow P$ such that $P \in \mathcal{P}$ and $\text{diam } f^{-1}(y) < \varepsilon$ for each $y \in P$. For a continuum X , we consider the following index $I(X)$ as follows: $I(X) \leq n$ if and only if there is a collection \mathcal{P} of compact polyhedra such that X is \mathcal{P} -like and $\text{rank } H^1(P) \leq n$ for each $P \in \mathcal{P}$.

3.6. Lemma. *For a continuum X , $I(X) \geq \text{rank } H^1(X)$.*

3.7. Corollary. *Let X be a 2-dimensional continuum. If X*

is \mathcal{P} -like and \mathcal{P} is a finite collection of compact polyhedra, then $\dim C(X) = \infty$.

Finally, we give the following problem.

Problem 2. Give the direct proof of 2.5 without using the Bing's result 2.4. The author believes that if Problem 2 is solved, Problem 1 would be solved by using the same methods.

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